A Note on the Exact Order of Systematic Sampled ARMA Models

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ABSTRACT

Assume ARMA(p',q')is a k-period sampled sequence from an ARMA(p,q)model, Brewer(1973) has show that p' = p and $q' = \left[\frac{p(k-1)+q}{k}\right]$,where[m] = the largest integer, M, satisfying $M \leq m$ [c.f. Helsinki (1984)], We find that the order q' does not always equal $\left[\frac{p(k-1)+q}{k}\right]$ May be any non-negative smaller imteger. Two counterexamples are given, furthermore, we provide a theorem to point out the exact value of q' directly via the information of any original ARMA(0,q) model

Keyword :

ARMA Model, systematic sampled model, Autocovariance function

1. Introduction :

Let Zt be an original seguence generated by a real-valued discrete ARMA(p,q)model :

$$Z_t = \sum_{i=1}^{p} \phi_i Z_{t-i} + \sum_{j=0}^{q} \theta_j e_{t-j} \qquad e_t \sim i\ i\ d(0\ \sigma_e^2)\ \forall_t \cdots\cdots (1.1)$$

Where

ϕ_i , θ_j $\sigma_e \in R$

$i = 1,2 \cdots\cdots ,p$

$j = 0,1,2 \cdots\cdots ,q$

$\theta_o = 1 \qquad \phi_p$, $\theta_q \neq 0$

for any positive integer k>1, If it is only every k th term is observed, We are concerned with the exact order of the sampled sequence $Y_t = Z_{kt}$

2. Definitions :

[Definition 1]

(i) If Zt is generated by an ARMA(p,o) model, We also say that Zt is generated by an AR(p) model.

(ii) If Zt is generated by (1.1) but $\phi_i = 0\ \ \forall_i = 1,2, \cdots\cdots ,p$ We say that Zt is generated by an MA(

q) model …………………………………………………………………………………………(2.1)

〔Definition 2〕

(i) If Zt is generated by (1.1) or (2.1) and $Y_t = Z_{kt}$ is generated by the following ARMA (p' , q') model :

$$Y_t = \sum_{j=1}^{p'} \phi'_i \; Y_{t-i} + \sum_{j=0}^{q'} \theta'_j \; e'_{t-j} \qquad e'_t \sim \text{i,i,d}(0,\ \sigma_e{}^2) \quad \forall_t \cdots\cdots (2.2)$$

where

$$\begin{cases} \phi'_t\ ,\ \theta_j \in R \\ \quad i = 1,2,\ldots,p' \\ \quad j = 0,1,2,\ldots,q' \\ \theta''_o = 1, \quad \phi'_{p'}\ ,\ \theta'_{q'} \neq 0 \\ \quad 0 < \sigma_{e'} \in R \end{cases}$$

Then We say that the model (2.2) is a k-period sampled sub-model of the model (1.1) or (2.1); or a systematic sampled sub-model of (1.1) or (2.1), and conversely, We say that the model (1.1) or (2.1) is the original of the model (2.1).

(ii) If Zt is generated by (1.1) or (2.1) and $Y_t = Z_{kt}$ is generated by (2.2), but $\phi'_i = 0 \quad \forall_i = 1,2, \cdots\cdots, p'$, i, e, ARMA (0, q') model we have the same definition as (i)

3. Properties :

〔 Theorem 1 〕

Let Zt be a real-valued sequence generated by an ARMA (2.0) model :

$$Z_t = \phi Z_{t-2} + e_t \qquad e_t \sim \text{iid N } (0,\ \sigma_e^2) \quad \forall t \cdots\cdots (3.1)$$

Then for any positive integer K the sampled sequence

$Y_t = Z_{kt}$ has the following properties :

(i) If k is an odd number

Then Yt is generated by an ARMA (2.0) model :

$$Y_t = \phi' Y_{t-2} + e'_t \qquad e'_t \sim \text{iid } (0,\ \sigma_e^2) \quad \forall t \cdots\cdots (3.2)$$

Where

$$\begin{cases} \begin{cases} \phi' = \phi^k \\ e'_t = \sqrt{\dfrac{1-\phi^{2k}}{1-\phi^2}} \quad e_t \quad \text{if } \phi \neq 1 \\ \sigma_{e'}^2 = \dfrac{1-\phi^{2k}}{1-\phi^2} \quad \sigma_e^2 \end{cases} \\ \begin{cases} \phi' = 1 \\ e'_t = \sqrt{k} \quad e_t \qquad \text{if} \phi = 1 \\ \sigma_{e'}^2 = k\,\sigma_e^2 \end{cases} \end{cases}$$

(ii) If k is an even number

then Yt is generated by an ARMA (2.1)model

$Y_t' = \phi' Y_{t-2} + e_t' + \theta' e_{t-1}'$

$e_t \sim \text{iid } N(0, \sigma_{e'}^2) \quad \forall t$ ……………………………………………… (3.3)

Where

$$\begin{cases} \begin{cases} \phi' = \phi^k \\ \theta' = \phi^{\frac{k}{2}} \\ e_t' = \sqrt{\dfrac{1-\phi^{2k}}{1-\phi^2}}\; e_t & \text{if } \phi \neq 1 \\ \sigma_{e'}^2 = \dfrac{1-\phi^{2k}}{1-\phi^2}\; \sigma_e^2 \end{cases} \\ \begin{cases} \phi' = 1 \\ \theta' = 1 \\ e_t' = \dfrac{k}{2} & \text{if } \phi = 1 \\ \sigma_{e'}^2 = \dfrac{k}{2}\; \sigma_e^2 \end{cases} \end{cases}$$

〔Proof〕

Let B be the lag operator (i.e, $BV_t = V_{t-1}$)

formula (3.1) may be rewriten as :

$(1-\phi B^2) Z_t = e_t$ ……………………………………………… (3.4)

and we have

$(1-\phi_k B^{2k}) Z_t = \left[\sum_{i=0}^{k-1} \phi^i B^{2i} \right] e_t \quad \forall t$ ……………………………… (3.5)

i,e, $Z_t - \phi^k Z_{t-2k} = \sum_{i=0}^{k-1} \phi^i e_{t-2i} \quad \forall t$ ……………………………… (3.6)

Peplace t by kt, We get :

$Z_{kt} - \phi^k Z_{k(t-2)} = \sum_{i=0}^{k-1} \phi_i e_{kt-2i} \quad \forall t$ ……………………………… (3.7)

Let $\begin{cases} Y_t = Z_{kt} \\ \varepsilon_t = \sum_{i=0}^{k-1} \phi^i e_{kt-2i} \end{cases}$

we can get the following systematic sampled sub-model of the ARMA (2.0) model in the from :

$Y_t = \phi^k Y_{t-2} + \varepsilon_t$ ……………………………………………… (3.9)

$E(\varepsilon_t) = E \sum_{i=0}^{k-1} \phi^i e_{kt-2i} = \sum_{i=0}^{k-1} \phi^i E(e_{kt-2i}) = 0$

Where

$\text{Var}(\varepsilon_t) = E(\varepsilon_t^2) = \sum_{i=0}^{k-1} \phi^{2i} E(e_{kt-2i}^2)$

$$= \sigma_e^2 \sum_{i=0}^{k-1} \phi^{2i} = \begin{cases} \sigma_e^2 \left[\dfrac{1-\phi^{2k}}{1-\phi^2} \right] & \phi \neq 1 \\ k\sigma_e^2 & \phi = 1 \end{cases}$$ ……………………………… (3.10)

Now consider the properties of $E(\varepsilon_t \ \varepsilon_{t-d})$ for $d \in N$, = If $d \geqq 2$

$$E(\varepsilon_t \varepsilon_t - d) = E(\varepsilon_t \varepsilon_t + d)$$

$$= E\left[\sum_{i=0}^{k-1} \phi^i e_{kt-2i} \left(\sum_{j=0}^{k-1} \phi^j e_{k(t+d)-2j} \right) \right]$$

$$= E\left[\sum_{i=0}^{k-1} \phi^i e_{kt-2i} \left(\sum_{j=0}^{k-1} \phi^j e_{kt+kd-2i} \right) \right] \cdots\cdots (3.11)$$

Consider the right hand side of (3.11) the largest footmark term in the first sequence is e_{kt}, and the smallest. footmark term in the secord sequence is $e_{kt+2+(d-2)k}$ Since $kt < kt + 2 + (d-2)k$ $\forall t \in Z$ we get $E(\varepsilon_t \ \varepsilon_{t-d}) = 0$

If $d = 1$

$$E(\varepsilon_t \ \varepsilon_{t-1}) = E(\varepsilon_t \ \varepsilon_{t+1})$$

$$= E\left[\sum_{i=0}^{k-1} \phi^i e_{kt-2i} \left(\sum_{j=0}^{k-1} \phi^j e_{k(t+1)-2j} \right) \right] \cdots\cdots (3.12)$$

Consider the e_i^2 term in (3.12)

(i) If k is an add number

There is no e_i^2 term in (3.12)for every i

Since $k_{t-2i} \neq k_{(t+1)-2j}$ $\forall t, i, j \in Z$ and $e_i \perp\!\!\!\perp e_j$ $\forall i \neq j$

we get $E(\varepsilon_t, \varepsilon_t - 1) = 0$ Let $\phi^k = \phi'$

$$\varepsilon_t = e_t'' = \begin{cases} \sqrt{\dfrac{1-\phi^{2k}}{1-\phi^2}}\, e_t & \text{if } \phi \neq 1 \\ k\ e_t & \text{if } \phi = 1 \end{cases}$$

Hence we oftain that the k-pericod systmatic sub-model of model (3.1) is the model (3.2)

(ii) If k is an even number

Consider the right hand side of (3.12)

The largest foot mark term in the first sequence is $e_{kt} = e_{k(t-1)-2(-\frac{k}{2})}$

The smallest foot mark term in the second sequence is $e_{k(t-1)-2(k-1)} = e_{kt-2(\frac{k}{2}-1)}$

We get all the following equal footmark term of the two sequences

$$e_{kt-2i} = e_{k(t-1)-2(\frac{k}{2}+i)} \cdots\cdots (3.13)$$

$$\forall_i = 0,1,2,\cdots\cdots,(\frac{k}{2} - 1)$$

Hence

$$E(\varepsilon_t \ \varepsilon_{t-1}) = E(\varepsilon_t \ \varepsilon_{t+1})$$

$$= E\left[\sum_{i=0}^{\frac{k}{2}-1} \phi^i \phi^{\frac{k}{2}+2i} e_{kt-2i} \ e_{k(t+1)-2(\frac{k}{2}+i)} \right]$$

$$= E\left[\sum_{i=0}^{\frac{k}{2}-1} \phi^{\frac{k}{2}+2i} e_{kt-2i}^2 \right]$$

$$= \sigma_e^2 \left[\sum_{i=0}^{\frac{k}{2}-1} \phi^{\frac{k}{2}+2i} \right]$$

$$=\begin{cases}\phi^{\frac{k}{2}}\left[\dfrac{1-\phi^k}{1-\phi^2}\right]\sigma_e^2 & \text{if } \phi\neq 1 \quad \cdots\cdots (3.14)\\ \dfrac{k}{2}\sigma_e^2 & \text{if } \phi=1\end{cases}$$

and we have

Case (a)

$$\phi\neq 1\Rightarrow E(\varepsilon_t\ \ \varepsilon_{t-d})=\begin{cases}\dfrac{1-\phi^{2k}}{1-\phi^2}\ \sigma_e^2 & d=0\\ \phi^{\frac{k}{2}}\left[\dfrac{1-\phi^k}{1-\phi^2}\right]\sigma_e^2 & d=1\\ 0 & d\geqq 2\end{cases}$$

$$\cdots\cdots (3.15)$$

Case (b)

$$\phi=1\Rightarrow E(\varepsilon_t\ \ \varepsilon_{t-d})=\begin{cases}k\sigma_e^2 & d=0\\ \dfrac{k}{2}\ \sigma_e^2 & d=1 \quad \cdots\cdots (3.16)\\ 0 & d\geqq 2\end{cases}$$

Therefore Zt is generated by a MA(1) model

Let $\varepsilon_t=a_t'+\theta'\ e'_{t-1}\quad e_t'\sim \text{iid } N(0,\ a^2_{e'})$ $\cdots\cdots$ (3.17)

We have

$$E(\varepsilon_t\ \ \varepsilon_{t-d})=\begin{cases}(1+\theta'^2)\sigma_e^2 & d=0\\ \theta'\sigma_e^2 & d=1 \quad \cdots\cdots (3.18)\\ 0 & d\geqq 2\end{cases}$$

for case (a)

compare (e.15) to (3.18) we get :

$$\sigma^2_{e'}\ \frac{1-\phi^k}{1-\phi^2}\sigma_e^2 \qquad \text{if } \theta'=\phi^{\frac{k}{2}} \qquad \phi\neq 1$$

$$\text{Let } e_t'=\sqrt{\frac{1-\phi^k}{1-\phi^2}}\ e_t$$

for case (b)

compare (3.16) to (3.18) we get :

$$\theta'=\phi'=1$$

$$e_t'=\sqrt{\frac{k}{2}}\,e_t \qquad \text{if } \phi=1$$

$$\sigma^2_{e'}=\frac{k}{2}\ \ \sigma_e^2$$

Therefore the proof is completed

Q.E.E

〔 Theorem 2 〕

Let Ze be al real-valued sequence generated by an ARMA (0,q) model

$$Z_t = \sum_{j=0}^{q} \theta_j e_{t-j} \qquad e_t \sim \text{iid } N(0, \sigma_e^2) \quad \cdots\cdots (3.19)$$

were

$$\theta_0 = 1 \quad \theta_j \in R \quad j = 1,2, \cdots\cdots ,q$$

$$0 < \sigma_e^2 < \infty$$

then for any positive integer k>1 and any given non-negative integer q' the sampled sequence $Y_t = Z_{kt}$ is generated by an MA (q') = ARMA (0, g'), if and only if the integer $g' \leqq \left[\frac{q}{k}\right]$ such that

$$\begin{cases} \sum_{j=0}^{q-kq'} \theta_j \theta_{j+kq} \neq 0 \\ \sum_{j=0}^{q-kd} \theta_j \theta_{j+kd} = 0 \end{cases} \quad \cdots\cdots (3.20)$$

where $\left[\frac{q}{k}\right]$ stands for the largest integer M satisfying $M \leqq q / k$

〔 proof 〕 Let

$$\theta_0 = 1 \quad \theta_j \in R \quad j \in N \text{ and } \quad 0 < \sigma_e^2 < \infty$$

$$e_t \sim \text{iid} \quad N(0\ a_e^2) \quad \forall t \in Z$$

then for any given non-negative integer q the MA-type sequence

$Z_t = \sum_{j=0}^{\infty} \theta_j e_{t-j}$ is generated by an ARMA (0,q) model (3.19)

if and only if

$$\begin{cases} \theta_q = \theta_o \theta_q \neq 0 \\ \theta_j > 0 \quad \forall_j > q \end{cases} \quad \cdots\cdots (3.21)$$

or if and only if q is the largest number of the integer d satisfying

$$\begin{cases} \sum_{j=0}^{q-d} \theta_j \theta_{j+d} \neq 0 & d = q \\ 0 & d > q \end{cases} \quad \cdots\cdots (3.22)$$

but we know that the sequence Zt is generated by an ARMA (0,q) if and only if the Autocovariance function of Zt is

$R_2(d) = E(Z_t Z_{t+d}) = E(Z_t Z_{t-d})$

$$= \begin{cases} \sum_{j=0}^{q-d} \theta_j \theta_{j+d} \sigma_e^2 & 0 \leqq d \leqq q \\ 0 & d > q \end{cases} \quad \cdots\cdots (3.23)$$

〔 See Fuller, w.A. (1976) 〕

Compare (3.23) to (3.22), therefore, we get that for any given non-negative integer q the sequence Zt is generated by an ARMA (0,q) model, if and only if the Autocovaricance function of Zt satisfies the following condition

$$\begin{cases} R_z(d) \neq 0 & d = q \\ R_z(d) = 0 & d > q \end{cases} \quad \cdots\cdots (3.24)$$

similarly we may analyse the case of systematic sampled sequences here :
for any given positive in teger $k > 1$

Since $Y_t = Z_{kt} = \sum_{j=0}^{q} \theta_j e_{kt-j}$

Autocovariance function of Yt :

$$R_y(d) = E(Y_t Y_{t+d}) = E(Z_{kt} Z_{kt-kd})$$

$$\begin{cases} \sum_{j=0}^{q-kd} \theta_j \theta_{j+kd} \sigma_e^2 & 0 \leqq kd \leqq q \\ 0 & kd > q \end{cases}$$

$$\text{or } R_y(d) = \begin{cases} \sum_{j=0}^{q-kd} \theta_j \theta_{j+kd} \sigma_e^2 & 0 \leqq d \leqq \left[\frac{q}{k} \right] \\ 0 & d > \left[\frac{q}{k} \right] \end{cases} \quad \cdots\cdots (3.25)$$

Where [m] stands for the largest integer M satisfying $M \leqq m$ form statement (3.24) We know that the sequence $Y_t = Z_{kt}$ is geneated by an ARMA $(0,[\frac{q}{k}])$ model if and only if

$$\begin{cases} R_y(d) \neq 0 & d = \left[\frac{q}{k} \right] \\ R_y(d) = 0 & d > \left[\frac{q}{k} \right] \end{cases} \quad \cdots\cdots (3.26)$$

In general for any given non-negative in teger q' the sampled sequence $Y_t = Z_{kt}$ is generated by an ARMA (0,q') model if and if

$$\begin{cases} R_y(d) \neq 0 & d = q' \\ R_y(d) = 0 & d > q' \end{cases} \quad \cdots\cdots (3.27)$$

In other words, for any given non-negative integer q', the sampled sequence $Y_t = Z_{kt}$ is generated by an ARMA (0,q') model if and only if the integer $q' \leqq [\frac{q}{k}]$ such that

$$\begin{cases} \sum_{j=0}^{q-kq} \theta_j \theta_j + q' \neq 0 \\ \sum_{j=0}^{q-kd} \theta_j \theta_{j+kd} = 0 \end{cases}$$

the proof is completed

Q.E.D.

4. Examples

[Example]

Consider the sub-models of the sampled sequence $Y_t = Z_{kt}$ sampling from some original ARMA (0, q) model (2.1) Let $q = 16$ $k = 5$ and $e_t \sim$ iid N (0.1) we have $[\frac{q}{k}] = 3$ $Y_t = Z_{5t}$ is generated by an ARMA (0.q') model if and only if the non-negative $q' \leqq 3$

such that

$$\begin{cases} \sum_{j=0}^{16-5d} \theta_j\, \theta_{j+5d} \neq 0 & d = q' \\ \sum_{j=0}^{16-5d} \theta_j\, \theta_{j+5d} = 0 & d > q' \end{cases} \quad \cdots\cdots (4.1)$$

(a) Let $Z_t = e_t + e_{t-4} + e_{t-8} + e_{t-12} + e_{t-16}$ be the original ARMA (0.16) model, then the sampled sequence $Y_t = Z_{5t}$ is not generated by an ARMA $(0,[\frac{q}{k}]) =$ ARMA (0.3) model It is gener ated by an ARMA (0,0) model i.e. $Y_t \sim$ iid N(0.5) Since the autocovariance function of Yt is

$$R_y(d) = \begin{cases} \sum_{j=0}^{q-kd} \theta_j\, \theta_{j+kd}\, \sigma_e^2 & 0 \leqq d \leqq \left[\frac{q}{k}\right] \\ 0 & d > \left[\frac{q}{k}\right] \end{cases}$$

$$= \begin{cases} 5 & d = 0 \\ 0 & d > 0 \end{cases} \quad \cdots\cdots (4.2)$$

(b) Let $Z_t = e_t + \frac{1}{2} e_{t-5} + \frac{1}{4} e_{t-7} + e_{t-7} + e_{t-16}$ be the original ARMA (0.16) model then the sampled sequence $Y_t = Z_{5t}$ is also not generaced by an ARMA $(0,[\frac{q}{k}]) =$ ARMA (0,3) model

$$\text{Since } R_y(d) = \begin{cases} 2.3125 & d = 0 \\ 0.5 & d = 1 \\ 0 & d > 1 \end{cases} \quad \cdots\cdots (4.3)$$

(c) Let $Z_t = e_t - e_{t-1} + \sum_{j=2}^{16} e_{t-j}$ be the original ARMA (0,16) complete model then the sampled sequence $Y_t = Z_{5t}$ is still not generated by an ARMA $(0,\frac{q}{k}) =$ ARMA (0,3), but an ARMA (0,2) model

$$\text{Since } R_y(d) = \begin{cases} 17 & d = 0 \\ 10 & d = 1 \\ 5 & d = 2 \\ 0 & d > 2 \end{cases} \quad \cdots\cdots (4.4)$$

(d) Let $Z_t = \sum_{j=0}^{16} e_{t-j}$ be the original ARMA (0,16) model then the sampled sequence $Y_t = Z_{5t}$ is exactly generated by an ARMA $(0,[\frac{q}{k}]) =$ ARMA (0,3), not geneated by an ARMA (0,4) model; Notice that 3 is the largest number of the integer d satisfying $d \leqq \frac{16}{5}$ hence $[\frac{16}{5}] = 3$ and 4 is the smallest number of the integer d satisfying $d \geqq \frac{16}{5}$ Hence $[\frac{16}{5}] \neq 4$ [c.f. Helsinki (1984)]

5. Conclusion :

Brewer (1973) has shown that if Zt is generated by an ARMA (p.q) model then $Y_t = Z_{kt}$ is generated by an ARMA $(p.[\frac{p(k-1)+q}{k}])$, In our [theorem 1] we get a counter-Example : if Zt is generated by an ARMA (2,0) model $Z_t = \phi\, Z_{t-2} + e_t$ when k is an odd number $Y_t = Z_{kt}$ is not generated by an ARMA $(2,\frac{2(k-1)+0}{k}) =$ ARMA (2,1), but an ARMA (2,0) model, it is true only if k is an even number, furthermore, we provide the [Theorem 2] to point out the exact order of the systematic sampled ARMA (0.q') model, and an applied example is given, by the way, there is a slip of the pen in Helsinki's paper (1984) on p.43 about the order 〔$\frac{p(k-1)+q}{k}$〕, where he say that "〔m〕 = the smallest integer M satisfying m ≦ M"

References

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系統抽樣ARMA模式之精確階數之研究

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摘要

假定 ARMA(p',q')爲抽自一 ARMA(p',q')模式之 K 期系統抽樣子數列，Brewer 於 1973 年證明了階數之值，具下述關係：p' ＝ p 且 q' ＝$\left[\frac{p(k-1)+q}{k}\right]$，其中高斯函數〔m〕表不大於 m 之最大整數，但筆者發現階數 q' 值並不恒爲$\left[\frac{p(k-1)+q}{k}\right]$除提出兩則反例以驗證上言外，進而並提出一個方便有用之定理，藉之可經由任意一個原始之 ARMA(0.q) 母模式的有關訊息，直接指出其抽樣子模式中階數 q' 值之精確值。

（註：本文曾在中華民國七十六年度統計學術研討會中發表摘要）。