A Note on the Exact Order of Systematic Sampled ARMA Models

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ABSTRACT

Assume ARMA(p',q')is a k-period sampled sequence from an ARMA(p,q)model, Brewer(1973) has shown that p' = p and q' = \left\lfloor \frac{pk - \frac{1}{2}}{k} + q \right\rfloor, where(m) = the largest integer, M, satisfying M \leq m (c.f. Helsinki (1984)). We find that the order q' does not always equal \left\lfloor \frac{pk - \frac{1}{2}}{k} + q \right\rfloor. May be any non-negative smaller integer. Two counterexamples are given, furthermore, we provide a theorem to point out the exact value of q' directly via the information of any original ARMA(0,q) model.

Keyword:
ARMA Model, systematic sampled model, Autocovariance function

1. Introduction:

Let Zt be an original sequence generated by a real-valued discrete ARMA(p,q) model:

\[ Z_t = \sum_{i=1}^{p} \phi_i Z_{t-i} + \sum_{j=0}^{q} \theta_j e_{t-j} \quad e_t \sim i.i.d(0, \sigma^2_e) \quad \forall t \]

Where

\[ \phi_i, \theta_j, \sigma_e \in \mathbb{R} \]

\[ i = 1,2, \ldots, p \]

\[ j = 0,1,2, \ldots, q \]

\[ \theta_0 = 1, \quad \phi_p, \theta_q \neq 0 \]

for any positive integer k>1. If it is only every k th term is observed, We are concerned with the exact order of the sampled sequence Y_t = Z_{kt}

2. Definitions:

(Definition 1)

(i) If Zt is generated by an ARMA(p,0) model, We also say that Zt is generated by an AR(p) model.

(ii) If Zt is generated by (1.1) but \( \phi_i = 0 \quad \forall i = 1,2, \ldots, p \) We say that Zt is generated by an MA(q) model.
q) model .................................................................................................................. (2.1)

(Definition 2)

(i) If $Z_t$ is generated by (1.1) or (2.1) and $Y_t = Z_{kt}$ is generated by the following ARMA ($p', q'$) model:

$$Y_t = \sum_{j=1}^{p'} \phi_j' Y_{t-j} + \sum_{j=0}^{q'} \theta_j' \epsilon_{t-j} + \epsilon_t \sim \text{i.i.d}(0, \sigma^2_e) \quad \forall t$$

(2.2)

where

$$\phi'_1, \ldots, \phi'_{p'} \in \mathbb{R}$$

$$i = 1, 2, \ldots, p'$$

$$j = 0, 1, 2, \ldots, q'$$

$$\theta_0'' = 1, \quad \phi''_0, \quad \theta''_0 \neq 0$$

$$0 < \sigma'_e \in \mathbb{R}$$

Then we say that the model (2.2) is a $k$-period sampled sub-model of the model (1.1) or (2.1); or a systematic sampled sub-model of (1.1) or (2.1), and conversely, we say that the model (1.1) or (2.1) is the original of the model (2.1).

(ii) If $Z_t$ is generated by (1.1) or (2.1) and $Y_t = Z_{kt}$ is generated by (2.2), but $\phi'_i = 0 \quad \forall i = 1, 2, \ldots, p'$, $\epsilon_t$, $e_t$, ARMA ($0, q'$) model we have the same definition as (i).

3. Properties:

[ Theorem 1 ]

Let $Z_t$ be a real-valued sequence generated by an ARMA (2.0) model:

$$Z_t = \phi Z_{t-2} + \epsilon_t + \epsilon_t \sim \text{i.i.d}(0, \sigma^2_e) \quad \forall t$$

(3.1)

Then for any positive integer $K$ the sampled sequence $Y_t = Z_{kt}$ has the following properties:

(i) If $k$ is an odd number

Then $Y_t$ is generated by an ARMA (2.0) model:

$$Y_t = \phi' Y_{t-2} + \epsilon'_t \quad \epsilon'_t \sim \text{i.i.d}(0, \sigma^2_e) \quad \forall t$$

(3.2)

Where

$$\phi' = \phi^k$$

$$\epsilon'_t = \frac{1 - \phi^k}{1 - \phi} \epsilon_t$$

if $\phi \neq 1$

$$\sigma'^2_e = \frac{1 - \phi^k}{1 - \phi^2} \sigma^2_e$$

$$\phi' = 1$$

$$\epsilon'_t = \sqrt{k} \epsilon_t$$

if $\phi = 1$

$$\sigma'^2_e = k \sigma^2_e$$

(ii) If $k$ is an even number
then \( Y_t \) is generated by an ARMA (2,1) model

\[
Y_t = \phi' Y_{t-2} + \epsilon_t' + \theta' \epsilon_{t-1}
\]

\( \epsilon_t \sim \text{iid } N(0, \sigma_e^2) \quad \forall t \)  

(3.3)

Where

\[
\left\{
\begin{array}{l}
\phi' = \phi^k \\
\theta' = \phi^{k-1} \\
\epsilon_t' = \frac{1 - \phi^k}{1 - \phi} \epsilon_t \quad \text{if } \phi \neq 1 \\
\sigma_e^2 = \frac{1 - \phi^k}{1 - \phi} \sigma_e^2 \\
\phi' = 1 \\
\theta' = 1 \\
\epsilon_t' = \frac{k}{2} \quad \text{if } \phi = 1 \\
\sigma_e^2 = \frac{k}{2} \sigma_e^2
\end{array}
\right.
\]

[ Proof ]

Let \( B \) be the lag operator (i.e., \( BV_t = V_{t-1} \))

formula (3.1) may be rewritten as:

\[
(1 - \phi B^2) Z_t = \epsilon_t \]

(3.4)

and we have

\[
(1 - \phi_k B^{2k}) Z_t = \left[ \sum_{i=0}^{k} \phi_i B^n \right] \epsilon_t \quad \forall t \]

(3.5)

i.e.,

\[
Z_t - \phi_k Z_{t-2k} = \sum_{i=0}^{k} \phi_i \epsilon_{t-2i} \quad \forall t
\]

(3.6)

Replace \( t \) by \( k_t \), We get:

\[
Z_{k_t} - \phi_k Z_{k_t-2k} = \sum_{i=0}^{k} \phi_i \epsilon_{k_t-2i} \quad \forall t
\]

(3.7)

Let

\[
Y_{k_t} = Z_{k_t}
\]

we can get the following systematic sampled sub-model of the ARMA (2.0) model in the form:

\[
Y_{k_t} = \phi_{k} Y_{k_t-2} + \epsilon_{k_t} \]

(3.9)

\[
E(\epsilon_{k_t}) = \sum_{i=0}^{k} \phi_i E(\epsilon_{k_t-2i}) = 0
\]

Where

\[
\text{Var}(\epsilon_{k_t}) = E(\epsilon_{k_t}^2) = \sum_{i=0}^{k} \phi_i E(\epsilon_{k_t-2i}^2)
\]

\[
= \sigma_e^2 \sum_{i=0}^{k} \phi_i = \left\{ \begin{array}{ll}
\sigma_e^2 \left[ 1 - \phi^k \right] & \phi \neq 1 \\
\frac{k}{\phi} \sigma_e^2 & \phi = 1
\end{array} \right.
\]

(3.10)
Now consider the properties of $E(\epsilon_t \epsilon_{t-d})$ for $d \in N_*$ if $d \geq 2$

$$E(\epsilon_t \epsilon_{t-d}) - d = E(\epsilon_t \epsilon_{t+d})$$

$$= E\left( \sum_{i=1}^{k} \phi^i \epsilon_{kt+i} \left( \sum_{j=1}^{k} \phi^j \epsilon_{kt+j} \right) \right)$$

$$= E\left( \sum_{i=1}^{k} \phi^i \epsilon_{kt+i} \left( \sum_{j=1}^{k} \phi^j \epsilon_{kt+j} \right) \right) \quad \text{(3.11)}$$

Consider the right hand side of (3.11) the largest footmark term in the first sequence is $e_{kt+i}$ and the smallest footmark term in the second sequence is $e_{kt+2(d-2)k}$

Since $kt < kt + 2 + (d - 2)k$

$\forall t \in Z$ we get $E(\epsilon_t \epsilon_{t-d}) = 0$

If $d = 1$

$$E(\epsilon_t \epsilon_{t-1}) = E(\epsilon_t \epsilon_{t+1})$$

$$= E\left( \sum_{i=1}^{k} \phi^i \epsilon_{kt+i} \left( \sum_{j=1}^{k} \phi^j \epsilon_{kt+j} \right) \right) \quad \text{(3.12)}$$

Consider the $e_t^2$ term in (3.12)

(i) If $k$ is an odd number

There is no $e_t^2$ term in (3.12) for every $i$

Since $k_{t-1} \neq k_{t+(t-1)z} \quad \forall t, i, j \in Z$ and $e_t, e_i \quad \forall i \neq j$

we get $E(\epsilon_t \epsilon_t - 1) = 0$ Let $\phi^k = \phi^t$

$$\epsilon_t = \epsilon_t'' = \begin{cases} \frac{1}{1 - \phi^k} \epsilon_t & \text{if } \phi^k \neq 1 \\ k \epsilon_t & \text{if } \phi^k = 1 \end{cases}$$

Hence we obtain that the $k$-periodic systematic sub-model of model (3.1) is the model (3.2)

(ii) If $k$ is an even number

Consider the right hand side of (3.12)

The largest footmark term in the first sequence is $e_{kt} = e_{kt_{(t-1)}z}$

The smallest footmark term in the second sequence is $e_{kt_{(t-1)}(2k-1)} = e_{kt_{2k-1}}$

We get all the following equal footmark term of the two sequences

$$e_{kt_{-1}} = e_{kt_{(t-1)}(2k-1)} \quad \text{(3.13)}$$

$\forall t = 0, 1, 2, \ldots, (\frac{k}{2} - 1)$

Hence

$$E(\epsilon_t \epsilon_{t-1}) = E(\epsilon_t \epsilon_{t+1})$$

$$= E\left( \sum_{i=1}^{k} \phi^i \epsilon_{kt+i} \epsilon_{kt+(t-1)(2k-1)} \right)$$

$$= E\left( \sum_{i=1}^{k} \phi^i \epsilon_{kt+i} \epsilon^3_{kt} \right)$$

$$= \sigma^2 \left[ \sum_{i=1}^{k} \phi^i \epsilon_{kt+i} \right]$$
\[
\sigma_e^2 = \begin{cases}
\frac{k}{2} \left[ \frac{1 - \phi^k}{1 - \phi^2} \right] \sigma_e^2 & \text{if } \phi \neq 1 \\
\frac{k}{2} \sigma_e^2 & \text{if } \phi = 1
\end{cases}
\]

and we have

Case (a)

\[\phi \neq 1 \Rightarrow \text{E}(\epsilon_t | \epsilon_{t-d}) = \begin{cases}
\frac{1 - \phi^k}{1 - \phi^2} \sigma_e^2 & d = 0 \\
\phi^\frac{k}{2} \left[ \frac{1 - \phi^k}{1 - \phi^2} \right] \sigma_e^2 & d = 1 \\
0 & d \geq 2
\end{cases}\]

(3.15)

Case (b)

\[\phi = 1 \Rightarrow \text{E}(\epsilon_t | \epsilon_{t-d}) = \begin{cases}
\frac{k \sigma_e^2}{2} & d = 0 \\
\frac{k}{2} \sigma_e^2 & d = 1 \\
0 & d \geq 2
\end{cases}\]

(3.16)

Therefore \(Z_t\) is generated by a MA(1) model

Let \(\epsilon_t = a_t + \theta' e'_{t-1}\) \(e'_t \sim \text{iid N}(0, \sigma_e^2)\)

(3.17)

We have

\[\text{E}(\epsilon_t | \epsilon_{t-d}) = \begin{cases}
(1 + \theta'^2) \sigma_e^2 & d = 0 \\
\theta' \sigma_e^2 & d = 1 \\
0 & d \geq 2
\end{cases}\]

(3.18)

for case (a)

compare (e.15) to (3.18) we get :

\[\sigma_e^2 = \frac{1 - \phi^k}{1 - \phi^2} \sigma_e^2 \text{ if } \theta' = \phi^\frac{k}{2} \neq 1\]

for case (b)

\[\theta' = \phi' = 1\]

\[e'_t = \frac{k}{2} \sigma_e^2 \text{ if } \phi = 1\]

Therefore the proof is completed

Q.E.E

[ Theorem 2 ]
Let $Z_t$ be a real-valued sequence generated by an ARMA $(0, q)$ model

$$Z_t \sim N(0, \sigma^2_e) \quad \text{and} \quad \epsilon_t \sim \text{iid } N(0, \sigma^2_e)$$

where

$$\theta_0 = 1 \quad \theta_j \in \mathbb{R} \quad j = 1, 2, \ldots, q \quad 0 < \sigma^2_e < \infty$$

then for any positive integer $k > 1$ and any given non-negative integer $q'$ the sampled sequence $Y_t = Z_{kt}$ is generated by an MA $(q') = \text{ARMA } (0, g')$, if and only if the integer $g' \leq \left\lfloor \frac{q}{k} \right\rfloor$

such that

$$\left\{ \begin{array}{c} \sum_{j=0}^{q'} \theta_j \theta_{q'k} \neq 0 \\ \sum_{j=0}^{q} \theta_j \theta_{qk} = 0 \end{array} \right.$$  \hspace{1cm} (3.20)

where $\left\lfloor \frac{q}{k} \right\rfloor$ stands for the largest integer $M$ satisfying $M \leq q / k$

[ proof ] Let

$$e_t \sim \text{iid } N(0, \sigma^2_e) \quad \forall t \in \mathbb{Z}$$

then for any given non-negative integer $q$ the MA-type sequence $Z_t = \sum_{j=0}^{\infty} \theta_j e_{t-j}$ is generated by an ARMA $(0, q)$ model (3.19)

if and only if

$$\left\{ \begin{array}{c} \theta_0 = 1 \quad \theta_j \in \mathbb{R} \quad j \in \mathbb{N} \quad \text{and} \quad 0 < \sigma^2_e < \infty \\ \epsilon_t \sim \text{iid } N(0, \sigma^2_e) \quad \forall t \in \mathbb{Z} \end{array} \right.$$  \hspace{1cm} (3.21)

or if and only if $q$ is the largest number of the integer $d$ satisfying

$$\left\{ \begin{array}{c} \sum_{j=0}^{d} \theta_j \theta_{qd} \neq 0 \quad d = q \\ 0 \quad d > q \end{array} \right.$$  \hspace{1cm} (3.22)

but we know that the sequence $Z_t$ is generated by an ARMA $(0, q)$ if and only if the Autocovariance function of $Z_t$ is

$$R_z(d) = \text{E}(Z_t Z_{t+d}) = \text{E}(Z_t Z_{t+d})$$

$$= \left\{ \begin{array}{c} \sum_{j=0}^{q} \theta_j \theta_{qd} \sigma^2_e \\ 0 \quad d > q \end{array} \right.$$  \hspace{1cm} (3.23)

[ See Fuller, W.A. (1976) ]

Compare (3.23) to (3.22), therefore, we get that for any given non-negative integer $q$ the sequence $Z_t$ is generated by an ARMA $(0, q)$ model, if and only if the Autocovariance function of $Z_t$ satisfies the following condition
\[
\begin{align*}
\left\{ \begin{array}{ll}
R_y(d) \neq 0 & d = q \\
R_y(d) = 0 & d > q
\end{array} \right. \quad \text{(3.24)}
\end{align*}
\]

Similarly, we may analyze the case of systematic sampled sequences here:

For any given positive integer \( k > 1 \),

Since \( Y_t = Z_{kt} = \sum_{j=0}^k \theta_j c_{kt-j} \),

Autocovariance function of \( Y_t \):

\[
R_y(d) = E(Y_t Y_{t+d}) = E(Z_{kt} Z_{kt+kd})
\]

\[
\begin{align*}
&\sum_{j=0}^{kd} \theta_j \theta_{kd} \sigma_t^2, \quad 0 \leq kd \leq q \\
&0, \quad kd > q
\end{align*}
\]

or

\[
R_y(d) = \left\{ \begin{array}{ll}
\sum_{j=0}^{kd} \theta_j \theta_{kd} \sigma_t^2, & 0 \leq d \leq \left\lfloor \frac{q}{k} \right\rfloor \\
0, & d > \left\lfloor \frac{q}{k} \right\rfloor
\end{array} \right. \quad \text{(3.25)}
\]

Where \( \lfloor m \rfloor \) stands for the largest integer \( M \) satisfying \( M \leq m \) in formal statement (3.24).

We know that the sequence \( Y_t = Z_{kt} \) is generated by an ARMA \((0, \frac{q}{k})\) model if and only if

\[
\left\{ \begin{array}{ll}
R_y(d) \neq 0 & d = \left\lfloor \frac{q}{k} \right\rfloor \\
R_y(d) = 0 & d > \left\lfloor \frac{q}{k} \right\rfloor
\end{array} \right. \quad \text{(3.26)}
\]

In general, for any given non-negative integer \( q' \), the sampled sequence \( Y_t = Z_{kt} \) is generated by an ARMA \((0, q')\) model if and only if

\[
\left\{ \begin{array}{ll}
R_y(d) \neq 0 & d = q' \\
R_y(d) = 0 & d > q'
\end{array} \right. \quad \text{(3.27)}
\]

In other words, for any given non-negative integer \( q' \), the sampled sequence \( Y_t = Z_{kt} \) is generated by an ARMA \((0, q')\) model if and only if the integer \( q' \leq \left\lfloor \frac{q}{k} \right\rfloor \) such that

\[
\begin{align*}
\sum_{j=0}^{kj} \theta_j \theta_{kj} + q' = 0 \\
\sum_{j=0}^{kd} \theta_j \theta_{kd} = 0
\end{align*}
\]

The proof is completed.

Q.E.D.

4. Examples

[ Example ]

Consider the sub-models of the sampled sequence \( Y_t = Z_{kt} \) sampling from some original ARMA \((0, q)\) model (2.1) Let \( q = 16 \), \( k = 5 \) and \( e_t \sim \text{iid N}(0,1) \) we have \( \left\lfloor \frac{q}{k} \right\rfloor = 3 \). \( Y_t = Z_{kt} \) is generated by an ARMA \((0, q')\) model if and only if the non-negative \( q' \leq 3 \)
such that

\[
\begin{cases}
\sum_{i=0}^{15} \theta_i \theta_{\mu d} \neq 0 & d = q' \\
\sum_{i=0}^{15} \theta_i \theta_{\mu d} = 0 & d > q'
\end{cases}
\]

\hspace{1cm} (4.1)

(a) Let \( Z_t = e_t + e_{t-4} + e_{t-8} + e_{t-12} + e_{t-16} \) be the original ARMA \((0,16)\) model, then the sampled sequence \( Y_t = Z_{5t} \) is not generated by an ARMA \((0,\left(\frac{16}{k}\right)) = \text{ARMA} \((0,3)\) model. It is generated by an ARMA \((0,0)\) model i.e. \( Y_t \sim \text{iid N}(0,0.5) \). Since the autocovariance function of \( Y_t \) is

\[
R_y(d) = \begin{cases}
\sum_{i=0}^{15} \theta_i \theta_{\mu d} \sigma^2 & 0 \leq d \leq \left[ \frac{-16}{k} \right] \\
0 & d > \left[ \frac{-16}{k} \right]
\end{cases}
\]

\hspace{1cm} (4.2)

(b) Let \( Z_t = e_t + \frac{1}{2} e_{t-5} + \frac{1}{4} e_{t-7} + e_{t-17} + e_{t-16} \) be the original ARMA \((0,16)\) model then the sampled sequence \( Y_t = Z_{5t} \) is also not generated by an ARMA \((0,\left(\frac{16}{k}\right)) = \text{ARMA} \((0,3)\) model.

Since \( R_y(d) = \begin{cases} 
2.3125 & d = 0 \\
0.5 & d = 1 \\
0 & d > 1
\end{cases} \)

\hspace{1cm} (4.3)

(c) Let \( Z_t = e_t - e_{t-1} + \frac{15}{12} e_{t-1} \) be the original ARMA \((0,16)\) complete model then the sampled sequence \( Y_t = Z_{5t} \) is still not generated by an ARMA \((0,\left(\frac{16}{k}\right)) = \text{ARMA} \((0,3)\), but an ARMA \((0,2)\) model.

Since \( R_y(d) = \begin{cases} 
17 & d = 0 \\
10 & d = 1 \\
5 & d = 2 \\
0 & d > 2
\end{cases} \)

\hspace{1cm} (4.4)

(d) Let \( Z_t = \sum_{i=0}^{15} e_{t-i} \) be the original ARMA \((0,16)\) model then the sampled sequence \( Y_t = Z_{5t} \) is exactly generated by an ARMA \((0,\left(\frac{16}{k}\right)) = \text{ARMA} \((0,3)\), not generated by an ARMA \((0,4)\) model. Notice that 3 is the largest number of the integer \( d \) satisfying \( d \leq \frac{16}{5} \) hence \( \left(-\frac{16}{5}\right) = 3 \) and 4 is the smallest number of the integer \( d \) satisfying \( d \geq \frac{16}{5} \) Hence \( \left(\frac{16}{5}\right) = 4 \).

(c.f. Helsinki (1984))

5. Conclusion:

Brewer (1973) has shown that if \( Z_t \) is generated by an ARMA \((p,q)\) model then \( Y_t = Z_{5t} \) is generated by an ARMA \((p,\left(\frac{p+1}{k}\right))\), In our (Theorem 1) we get a counter-Example : if \( Z_t \) is generated by an ARMA \((2,0)\) model \( Z_t = \phi Z_{t-2} + e_t \) when \( k \) is an odd number \( Y_t = Z_{5t} \) is not generated by an ARMA \((2,2\left(\frac{1}{k}\right)) = \text{ARMA} \((2,1)\), but an ARMA \((2,0)\) model, it is true only if \( k \) is an even number, furthermore, we provide the (Theorem 2) to point out the exact order of the systematic sampled ARMA \((0,q')\) model, and an applied example is given, by the way, there is a slip of
the pen in Helsinki's paper (1984) on p.43 about the order \( \left( \frac{p^{(k-1)+a}}{k} \right) \), where he say that \( \lceil m \rceil = \text{the smallest integer } M \text{ satisfying } m \leq M \).

References

系統抽樣 ARMA 模式之精確階數之研究

摘要

假定 ARMA(\(p',q'\)) 為自 - ARMA(\(p',q'\)) 模式之 K 期系統抽樣子數列，Breuer 於 1973 年證明了階數之值，具下述關係：\(p' = p\) 且 \(q' = \left\lfloor \frac{p(q+1)+a}{k} \right\rfloor\)，其中高斯函數 \(\text{G}(m)\) 表不不大於 \(m\) 之最大整數，但筆者發現階數 \(q'\) 值並不恆為 \(\left\lfloor \frac{p(q+1)+a}{k} \right\rfloor\) 除提出兩則反例以驗證上言外，進而並提出一個方便有用之定理，藉之可經由任意一個原始之 ARMA(0,q) 母模式的有關訊息，直接指出其抽樣子模式中階數 \(q'\) 值之精確值。

（註：本文曾在中華民國七十六年度統計學術研討會中發表摘要）。